Cobb-douglas concavity

Given the following function:

$$y = (x_1)^\alpha (x_2)^\beta$$

where α, β are positive and $\alpha, \beta < 1$.

1. Show it is quasi-concave.
2. Show that if $\alpha + \beta > 1$ then the Cobb-Douglas function is not concave (which shows that not all quasi-concave functions are concave).

Solutions

1. Taking the first derivatives

$$f_1 = \alpha x_1^{\alpha-1} x_2^{\beta}$$
$$f_2 = \beta x_1^{\alpha} x_2^{\beta-1}$$

Taking the second derivatives

$$f_{11} = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{\beta} < 0$$
$$f_{22} = \beta(\beta-1) x_1^{\alpha} x_2^{\beta-2} < 0$$

$$f_{12} = f_{21} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} > 0$$

The quasiconcavity condition can be determined by the sign of the determinant of a bordered hessian matrix. The bordered Hessian is given by:

$$\bar{H} = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{bmatrix}$$

And its determinant is:

$$\det(\bar{H}) = -f_1(f_1 f_{22} - f_2 f_{21}) + f_2(f_1 f_{12} - f_{11} f_2) = -f_1^2 f_{22} + f_2 f_1 f_{21} + f_2 f_1 f_{12} - f_2^2 f_{11}$$
$$\det(\bar{H}) = -f_1^2 f_{22} + 2 f_2 f_1 f_{21} - f_2^2 f_{11}$$

For a function to be quasi-concave, the determinant of the bordered Hessian matrix should be positive:

$$\det(\bar{H}) = -f_1^2 f_{22} + 2 f_2 f_1 f_{21} - f_2^2 f_{11} > 0$$

Which is true in this case since all those terms are positive.

2. A function is concave if and only if the hessian is negative semidefinite, the hessian matrix is:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

And is negative semidefinite if

$$f_{11} \leq 0$$

And

$$\det(H) = f_{11} f_{22} - f_{12}^2 \geq 0$$

Replacing with the values from the cobb-douglas:

$$\alpha(\alpha-1) x_1^{\alpha-2} x_2^{\beta} \leq 0$$

But:

$$f_{11} f_{22} - f_{12}^2 = \alpha\beta(\alpha-1)(\beta-1) x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha^2\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$
$$f_{11} f_{22} - f_{12}^2 = \alpha^2\beta(\beta-1) x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha\beta(\beta-1) x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha^2\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$
$$f_{11} f_{22} - f_{12}^2 = \alpha^2\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha^2\beta x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2} + \alpha\beta x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha^2\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$$f_{11} f_{22} - f_{12}^2 = -\alpha^2\beta x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2} + \alpha\beta x_1^{2\alpha-2} x_2^{2\beta-2} = \alpha\beta x_1^{2\alpha-2} x_2^{2\beta-2}(1-\alpha-\beta)$$

which is negative for $\beta + \alpha > 1$.